

Holographic Interpolation between a and F

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An interpolating function \tilde{F} between the a -anomaly coefficient in even dimensions and the free energy on an odd-dimensional sphere has been proposed recently and is conjectured to monotonically decrease along any renormalization group flow in continuous dimension d . We examine \tilde{F} in the large- N CFT's in d dimensions holographically described by the Einstein-Hilbert gravity in the AdS_{d+1} space. We show that \tilde{F} is a smooth function of d and correctly interpolates the a coefficients and the free energies. The monotonicity of \tilde{F} along an RG flow follows from the analytic continuation of the holographic c -theorem to continuous d , which completes the proof of the conjecture.

I. INTRODUCTION

A measure of degrees of freedom in a quantum field theory (QFT) remains to be elucidated in arbitrary d dimensions. Physically, it decreases monotonically as the energy scale is lowered because of the decoupling of massive particles. Implementation of such a measure in any QFT in diverse dimensions is intriguing and desirable to characterize the behavior under a renormalization group (RG) flow.

For even d , the conformal anomaly in the stress-energy tensor [1]

$$\langle T_\mu^\mu \rangle = \frac{(-1)^{\frac{d}{2}+1}}{2} a E_d + \sum_i b_i I_i, \quad (1)$$

defines the unique a coefficient for the Euler density E_d and several b_i coefficients for the Weyl invariants I_i labeled by an integer i . The a coefficients are believed to be monotonically decreasing along any RG flow, namely the value a_{UV} at the ultra-violet (UV) fixed point is equal or greater than that a_{IR} at the infra-red (IR) fixed point, $a_{\text{UV}} \geq a_{\text{IR}}$. This statement was established in two dimensions by the Zamolodchikov's c -theorem [2] and in four dimensions by the a -theorem [3–5]. On the other hand, the F -theorem asserts that the free energy, $F \equiv (-1)^{\frac{d-1}{2}} \log Z_{S^d}$, defined by the conformal invariant partition function Z_{S^d} on S^d of radius R , decreases under any RG flow in odd dimensions [6, 7]. A proof for $d = 3$ was presented by [8] through the relation of the free energy to the entanglement entropy S across an entangling surface S^{d-2} of radius R in $\mathbb{R}^{1,d-1}$ [9]

$$F = (-1)^{\frac{d-1}{2}} S, \quad (2)$$

that holds for odd d up to UV divergences.

These two proposals look quite different at first sight, but share the fact that both the a coefficient and the free energy can be read off on S^d ; the former arises from the integration of the trace of the stress-energy tensor (1) and the latter from the partition function. To interpolate

between the a coefficient and the free energy, Giombi and Klebanov define a new function [10]

$$\tilde{F} \equiv \sin\left(\frac{\pi d}{2}\right) \log Z_{S^d}, \quad (3)$$

which correctly reduces to the free energies for odd d . They show as d approaches to even integers [11] (see also [12] as a related work)

$$\tilde{F} = \frac{\pi}{2} a. \quad (4)$$

Note that the partition function Z_{S^d} used in (3) is conformal invariant and UV divergent for even d . The relation (4) follows from the fact that the conformal invariant partition function in $d = 2n + \epsilon$ dimensions behaves as $\log Z_{S^d} = (-1)^{\frac{d}{2}} \frac{a}{2\epsilon} + O(1)$ for small ϵ . This is because one has to add a local counter term

$$I_{\text{c.t.}} = (-1)^{\frac{d}{2}+1} \frac{a}{2\epsilon} \int_{S^d} d^d x \sqrt{g} E_{2n}, \quad (5)$$

to the partition function to obtain the renormalized partition function $\log Z_{S^d}^{(\text{ren})} = \log Z_{S^d} + I_{\text{c.t.}}$, reproducing the conformal anomaly $\log Z_{S^{2n}}^{(\text{ren})} = (-1)^{n+1} a \log R$ on S^{2n} of radius R in $\epsilon \rightarrow 0$ limit.

The function \tilde{F} is also defined for non-integer d and therefore smoothly interpolates between the a coefficients in even dimensions and the free energies in odd dimensions. They conjecture that \tilde{F} is positive and decreases along any RG flow in arbitrary d dimensions, based on several examples including a double-trace deformation of the large- N conformal field theory (CFT). We will call their proposal the \tilde{F} -theorem.

In this letter, we provide a further evidence to the \tilde{F} -theorem from the holographic viewpoint. To this end, we take advantage of the relation (2) and calculate the holographic entanglement entropy [13, 14] across a sphere S^{d-2} in the Einstein-Hilbert gravity on the AdS_{d+1} space. We perform the dimensional regularization in the bulk and obtain the analytic result of \tilde{F} that is a positive and smooth function of dimension d . We show

that the equality (4) holds for even d and furthermore prove the \tilde{F} -theorem that follows from the holographic c -theorem [15–18] assuming the dimensional continuation of the null energy condition.

II. HOLOGRAPHIC PROOF OF THE \tilde{F} -THEOREM

We will evaluate \tilde{F} with the relation (2) between the free energy on S^d and the entanglement entropy across S^{d-2} . The latter can be holographically calculated by the Ryu-Takayanagi formula in the Einstein-Hilbert gravity [13, 14]

$$S = \frac{\text{Area}(\gamma)}{4G_N^{(d+1)}}, \quad (6)$$

where $G_N^{(d+1)}$ is the Newton constant, and γ stands for the $(d-1)$ -dimensional minimal surface in the AdS_{d+1} space, whose boundary is the entangling surface S^{d-2} . Since the boundary of the AdS_{d+1} space is the flat space $\mathbb{R}^{1,d-1}$, we will use the Poincaré coordinates

$$ds^2 = L^2 \frac{dz^2 - dt^2 + dr^2 + r^2 d\Omega_{d-2}^2}{z^2}, \quad (7)$$

where L is the AdS radius. The entangling surface is located at $t = 0$ and $r = R$ at the boundary $z = 0$. In these coordinates, the minimal surface γ in the bulk is a hemi-hypersphere satisfying $r^2 + z^2 = R^2$ [13, 14]. This solution leads the entanglement entropy across S^{d-2}

$$S = \frac{1}{4G_N^{(d+1)}} L^{d-1} \text{Vol}(S^{d-2}) \int_{\epsilon/R}^1 dy \frac{(1-y^2)^{\frac{d-3}{2}}}{y^{d-1}}, \quad (8)$$

where we introduced a small cutoff at $z = \epsilon$ to regularize the UV divergence and $\text{Vol}(S^{d-2})$ is the volume of a unit $(d-2)$ -dimensional round sphere. Expanding the integrand with respect to y and performing the integration, one obtains the UV divergent parts of the entanglement entropy. We, however, want to employ the dimensional regularization instead of putting the UV cutoff at $z = \epsilon$ for our purpose. So we take $\epsilon = 0$ and carry out the integral in the range $1 < d < 2$, that yields

$$S = \frac{L^{d-1}}{4G_N^{(d+1)}} \pi^{\frac{d}{2}-1} \Gamma\left(1 - \frac{d}{2}\right). \quad (9)$$

Then we analytically continue d to any real value. It is clear that there are poles at even d in the entanglement entropy (9) corresponding to the conformal anomalies. Finally, using the relations (2) and (3), and the formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, we obtain \tilde{F} in the holographic theories

$$\tilde{F} = \frac{L^{d-1}}{4G_N^{(d+1)}} \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \quad (10)$$

This is manifestly a positive and smooth function of dimension d without poles at even d .

Now let us extrapolate the holographic values of \tilde{F} to even dimensions and see if the relation (4) holds. The a coefficients holographically computed in the Einstein-Hilbert gravity are known to be [17–20]

$$a = \frac{L^{d-1}}{2\pi G_N^{(d+1)}} \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \quad (11)$$

Combining it with (10), we confirm the relation (4) between \tilde{F} and a . Moreover, imposing the null energy condition in the bulk, the holographic c -theorem states that the a coefficient given by (11) satisfies the monotonicity, $a_{\text{UV}} \geq a_{\text{IR}}$, for positive integer d [15–18]. Assuming the analytic continuation of dimension d in the gravity, the holographic c -theorem holds for $d \geq 1$ [21] which assures the \tilde{F} -theorem due to the relation (4).

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